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# THE REFLECTION OF NORMAL MODES FROM A SEMI-INFINITE SYSTEM OF BARRIERS IN A WAVEGUIDE†

V. D. LUK'YANOV and G. L. NIKITIN
St Petersburg
e-mail: lvv@vl2771.spb.edu

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The steady reflection of acoustic waves from a semi-infinite system of equally spaced barriers in a waveguide is investigated. Each barrier consists of a rigid diaphragm with an aperture, which is covered with a rigid moving valve. The valve is coupled elastically to the diaphragm and is capable of performing small oscillations. An exact analytical solution is obtained for the case of a waveguide of arbitrary cross-section, and results of a numerical investigation of the reflection coefficients of normal modes as a function of the frequency of the incident field are presented for the case of the plane problem. For comparison, analytical expressions and graphs of the frequency dependences for the reflection and transmission coefficients of normal modes through one and two similar barriers in the waveguide are given. 

The propagation of different kinds of modes in periodic one-, two- and three-dimensional structures, consisting of either point masses, connected by an elastic coupling, or of their electric or acoustic equivalents, has been considered previously in [1, 2].

As is well known, when investigating the propagation of sound in acoustic devices, (sound conductors and acoustic filters), made in the form of tubes (waveguides) with different couplings (cavities, expanders, dampers, etc.), these components are regarded as a non-uniformity with lumped parameters. The acoustic properties of the non-uniformities are characterized by the acoustic mass and the acoustic compliance [3]. The simplest mechanical model, which can be used to describe the propagation and scattering of sound by non-uniformities in such acoustic devices, is a piston, attached by an elastic spring. The oscillations of a single piston, completely covering the channel of a circular pipeline were considered in [4]. The scattering of waves by a piston, situated at the junction of two waveguides with different rigid and soft walls was investigated in [5].

Below we obtain exact solution of the problem of the reflection of normal modes of an acoustic waveguide from one or two obstacles, and also from a semi-infinite system of equally spaced obstacles in them. The obstacles are rigid screens with apertures, which are covered by moving pistons, supported by elastic springs (Fig. 1). The problem is a mathematical model for investigating the problem of the reflection of sound from acoustic filters with a large number of components.

## 1. FORMULATION OF THE PROBLEM

Consider an acoustic waveguide, which is a cylindrical body filled with an ideal compressible medium and having an arbitrary cross-section $G$, and a boundary $\partial G$. The geometry of the problem and the choice of the system of coordinates are shown in Fig. 1. The $z$ axis is parallel to the generatrix of the cylindrical body. The acoustic pressure $P(M, z)$ in the medium filling the waveguide satisfies the homogeneous Helmholtz equation

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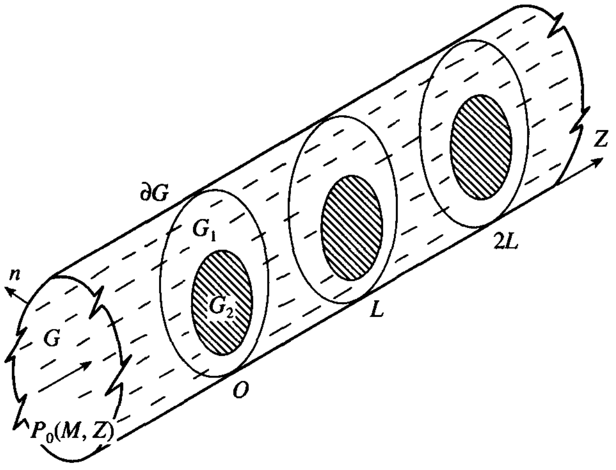


Fig. 1

$$\left(\Delta_{\perp}+\frac{\partial^2}{\partial z^2}\right)P(M,z)+k^2P(M,z) = 0, \quad k = \frac{\omega}{c}$$

The point $M$ is situated in cross-section of the waveguide with coordinate $z$, $\Delta_{\perp}$ is the two-dimensional Laplace operator in the cross-section of the waveguide, $k$ is the wave number, $c$ is the velocity of sound in the medium and $\omega$ is the angular frequency. The factor $\exp(-i\omega t)$, which specifies the harmonic dependence of the wave processes on the time $t$, is omitted here and everywhere henceforth in the expressions for the pressure.

The Neumann boundary condition

$$\frac{\partial P}{\partial n}(M,z) = 0 \quad \text{when} \quad M \in \partial G, \quad -\infty < z < \infty$$

where $n$ is the normal to the side surface of the waveguide, is satisfied on the absolutely rigid wall of the waveguide.

The steady scattering of normal modes by a semi-infinite system of transverse barriers in the waveguide is investigated. The barriers are situated in cross-sections of the waveguide $z = jL$; here and everywhere henceforth $j = 0, 1, 2, \ldots$ . Each barrier consists of a rigid diaphragm, occupying the region $G_1$. The aperture in the diaphragm is covered by a rigid moving valve of arbitrary shape, which occupies the region $G_2$, where $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \varnothing$. The Neumann boundary condition

$$\frac{\partial P}{\partial z}(M, jL \pm 0) = 0 \text{ when } M \in G_1 \tag{1.1}$$

is satisfied on the surface of the diaphragm.

The valve of mass $m$ is supported by an elastic spring of stiffness $g$ and performs small oscillations about the equilibrium position. The equation of motion of the valve, taking into account the fact that the wave processes depend harmonically on time, can be written in the form

$$(g-m\omega^2)U_j = \iint\limits_{G_2}(P(M, jL-0)-P(M, jL+0))dS \tag{1.2}$$

where $U_j$ is the displacement of the valve, situated in the section $z = jL$, from the equilibrium position.

The kinematic condition of the contact between the valve and the medium, which consists of equating the displacement of the valve and the component of the displacement of the medium in the region of the valve, normal to it, has the form

$$U_j = \frac{1}{\rho\omega^2}\frac{\partial P}{\partial z}(M, jL \pm 0) = 0 \quad \text{when} \quad M \in G_2 \tag{1.3}$$

The acoustic field scattered by the barriers must satisfy the limiting absorption principle, and in the neighbourhood of the points where the diaphragms are joined to the walls of the waveguide and the valves it must satisfy the Meixner conditions.

We will choose as the source of the excitation of oscillations in the waveguide with the barriers a propagating normal mode $Q_0(M, z)$, incident on the barriers from the side of negative values of $z$.

## 2. THE RELATION BETWEEN THE VALUE OF THE VALVE DISPLACEMENT AND THE AMPLITUDES OF THE NORMAL MODES

In an acoustic waveguide with rigid walls without barriers there is a set of solutions of the homogeneous problem, called normal modes

$$p_s^{\pm}(M, z) = A\varphi_s(M)\exp(\pm i\gamma_s z), \quad s = 0, 1, 2, \ldots \tag{2.1}$$

The constant $A$ has the dimension of pressure. The functions $\varphi_s(M)$, which describe the acoustic pressure distribution in a normal mode in a cross-section of the waveguide, are eigenfunctions, while $\mu_s = \lambda_s^2$ are eigenvalues of the Neumann problem for the Laplace operator $\Delta_\perp$ in the region $G$ with boundary $\partial G$

$$\Delta_\perp\varphi_s(M) = \mu_s\varphi_s(M) \text{ when } M \in G \text{ and } \frac{\partial\varphi_s}{\partial n}(M) = 0 \text{ when } M \in \partial G.$$

The eigenvalues $\mu_2 \le 0$ are numbered in decreasing order, and the eigenfunctions $\varphi_s(M)$ form an orthogonal basis in the space $L_2(G)$.

The propagation constant $\gamma_s$ of the normal mode $p_s^{\pm}(M, z)$ is connected with the eigenvalue $\mu_s$ by the relation $\gamma_s^2 = k^2 + \mu_s$. The frequencies $\tilde{\omega}_s$, at which the propagation constant $\gamma_s$ vanishes, are called the onset frequencies of the normal modes. At frequencies higher than the onset frequencies when $k > \lambda_s$, the normal mode $p_s^+(M, z)$ is a propagating mode and transfers oscillatory energy in the positive direction of the $Oz$ axis, while the mode $p_s^-(M, z)$ propagates in the negative direction.

The normalization of the eigenfunctions $\varphi_s(M)$ for propagating modes is chosen in the form

$$\frac{\gamma_s}{\sigma k}\iint_G \varphi_s^2(M)dS = 1 \tag{2.2}$$

where $\sigma$ is the waveguide cross-section area chosen so that all the propagating normal modes $p_s^+(M, z)$ transfer the same energy flux, averaged over a period,

$$E = \pm\frac{1}{2\rho\omega}\mathrm{Im}\iint_G \overline{p_s^{\pm}(M, z)}\frac{\partial p_s^{\pm}}{\partial z}(M, z)dS = \frac{A^2}{2\rho c}$$

through the waveguide cross-section. Here we take the indices either plus or minus simultaneously, and the bar denotes complex conjugation.

In particular, it follows from condition (2.2) that the eigenfunction $\varphi_0(M) = 1$, which specifies a uniform pressure distribution in the waveguide cross-section, corresponds to the eigenvalue $\mu_0 = 0$ in the region $G$. This wave is called a piston wave, and it is propagating at all excitation frequencies.

For non-uniform waveguide modes the "normalization" condition is also chosen to be relation (2.2).

Note that in problems where separation of the variables in the region $G$ occurs, the number of an eigenfunction $s$ must be understood as the multi-index $s = (s_1, s_2)$.

We will characterize the displacement of the valve, situated in the cross-section $z = jL$, by the dimensionless quantity $u_j$, and connected with the displacement of this valve $U_j$ by the relation

$$U_j = \frac{Au_j}{-i\omega\rho c} \tag{2.3}$$

Suppose one of the propagating normal modes with number $s$

$$Q_0(M, z) = p_s^+(M, z) \tag{2.4}$$

is incident on the barrier from the side of negative values of $z$.

The total acoustic field in the waveguide to the left of the first barrier will be sought in the form of an expansion in normal modes (everywhere henceforth summation is carried out over $n$ from zero to infinity)

$$P(M, z) = Q_0(M, z) + \sum r_{ns} p_s^-(M, z) \tag{2.5}$$

The series in formula (2.5) describes the reflection of the field from the first barrier in the waveguide, situated in the cross-section $z = 0$, $r_{ss}$ is the reflection coefficient of the mode $p_s^+(M, z)$, and $r_{ns}$ is the transformation factor of the incident wave into the $n$th reflected normal mode ($n \neq s$).

We will show that the required quantities $r_{ns}$ in (2.5) are uniquely defined by the amplitude of the oscillation of the valve $U_0$. To do this, making use of the characteristic function $\chi(M)$ of the region $G_2$ ($\chi(M) = 1$ if $M \in G_2$ and $\chi(M) = 0$ if $M \in G_1$, we rewrite boundary conditions (1.1) and (1.3) in the form of the single condition

$$\chi(M)U_0 = \frac{1}{\rho\omega^2}\frac{\partial P}{\partial z}(M, -0) \quad \text{when} \quad M \in G \tag{2.6}$$

Condition (2.6), taking (2.5) into account, leads to the equation

$$\rho\omega^2 U_0 = i\gamma_s p_s^+(M, 0) + \sum(-i\gamma_n) r_{ns} p_n^-(M, 0) \tag{2.7}$$

Multiplying (2.7) by the function $\varphi_{l}(M)/(\sigma k)$ with $n' = 0, 1, 2, \ldots$ and integrating the equation obtained over the whole surface of the barrier $G$, taking the normalization (2.2) and the relation (2.3) into account, we obtain

$$r_{ns} = \delta_{ns} - \eta_n u_0; \quad \eta_n = \frac{1}{\sigma}\iint_G \chi(M)\varphi_n(M)dS = \frac{1}{\sigma}\iint_{G_2} \varphi_n(M)dS \tag{2.8}$$

where $\delta_{ns}$ is the Kronecker delta and $\eta_n$ are the Fourier coefficients of the characteristic function $\chi(M)$: $\chi(M) = \sum \eta_n \varphi_n(M)$. Note that $\eta_0 = \sigma_2/\sigma$, where $\sigma_2$ is the area of the region $G_2$ which the valve occupies.

The acoustic pressure $P_j(M, z)$ in the section of the waveguide between neighbouring barriers, situated in the cross-sections $z = jL$ and $z = (J + 1)L$, will be sought in the form of an expansion in series in standing waves

$$P_j(M, z) = \sum a_n^{(j)} q_n^{(s)}(M, z - z_c^{(j)}) + b_n^{(j)} q_n^{(a)}(M, z - z_c^{(j)}) \tag{2.9}$$

where

$$q_n^{(s)}(M, z) = (p_n^+(M, z) + p_n^-(M, z))/2 = A\varphi_n(M)\cos(\gamma_n z)$$

$$q_n^{(a)}(M, z) = i(p_n^+(M, z) - p_n^-(M, z))/2 = A\varphi_n(M)\sin(\gamma_n z)$$

where $a_n^{(j)}$ and $b_n^{(j)}$ are the amplitudes of the standing waves, $q_n^{(s)}(M, z)$ is the standing wave in the cavity, symmetric with respect to the cross-section $z = z_c^{(j)} = \left(j + \frac{1}{2}\right)L$ and $q_n^{(a)}(M, z)$ and is the standing wave antisymmetric with respect to this cross-section.

From the boundary condition (2.6) for each valve, taking (2.9) into account, we obtain

$$\rho\omega^2 U_{j+k}\chi(M) = \sum \gamma_n(-1)^k\left(a_n^{(j)} q_n^{(a)}\left(M, \frac{L}{2}\right) + b_n^{(j)} q_n^{(s)}(M, L/2)\right), \quad k = 0, 1 \tag{2.10}$$

As when deriving formula (2.8), from Eqs (2.10) we obtain the following system of linear algebraic equations

$$a_n^{(j)}\sin(\gamma_n L/2) + b_n^{(j)}\cos(\gamma_n L/2) = i\eta_n u_j$$

$$-a_n^{(j)}\sin(\gamma_n L/2) + b_n^{(j)}\cos(\gamma_n L/2) = i\eta_n u_{j+1}$$

from which we find the quantities $a_n^{(j)}$ and $b_n^{(j)}$. If $\sin(\gamma_n L) \neq 0$, we obtain

$$a_n^{(j)} = \frac{i\eta_n}{2\sin(\gamma_n L/2)}(u_j - u_{j+1}) \tag{2.11}$$

$$b_n^{(j)} = \frac{i\eta_n}{2\cos(\gamma_n L/2)}(u_j + u_{j+1}) \tag{2.12}$$

The case when the displacements of the values are equal $u_j = u_{j+1}$ corresponds to the case $\sin(\gamma_n L/2) = 0$, in which case $a_n^{(j)} = 0$, and quantity $b_n^{(j)}$ is given by formula (2.12). When $\cos(\gamma_n L/2) = 0$ we have $u_j = -u_{j+1}$, and then $b_n^{(j)} = 0$, while the quantity $a_n^{(j)}$ is found from (2.11).

## 3. THE SCATTERING OF NORMAL MODES BY A SEMI-INFINITE SYSTEM OF BARRIERS IN A WAVEGUIDE

For a final solution of the problem of the scattering of normal modes by barriers in a waveguide, we must determine the amplitudes of the displacements of the valves. The required amplitudes are obtained using their equations of motion (1.2), whence we obtain an infinite system of linear algebraic equations for these amplitudes.

The equation of motion of the valve in the cross-section $z = 0$ has the form

$$(g - m\omega^2)U_0 = \iint\limits_{G_2} (p_s^+(M,0) + \sum r_{ns}p_n^-(M,0))dS -$$

$$-\iint\limits_{G_2} (\sum a_n^{(0)} q_n^{(s)}(M, L/2) - b_n^{(0)} q_n^{(a)}(M, L/2))dS \tag{3.1}$$

Integrating on the right-hand side of Eq. (3.1) and taking into account relation (2.3) and formulae (2.11) and (2.12), we have

$$u_0\left(Z_* + \sum Z_n + \frac{1}{2}\sum(Z_n^+ + Z_n^-)\right) = \frac{u_1}{2}\sum(Z_n^+ + Z_n^-) + 2\eta_s \tag{3.2}$$

$$Z_* = -\frac{i\omega m_0}{\rho c}\left(1 - \frac{\omega_0^2}{\omega^2}\right), \quad Z_n = \frac{k\eta_n^2}{\gamma_n}, \quad \omega_0 = \sqrt{\frac{g}{m}}, \quad m_0 = \frac{m}{\sigma_2}$$

$$Z_n^+ = iZ_n \operatorname{ctg}(\gamma_n L/2), \quad Z_n^- = -iZ_n \operatorname{tg}(\gamma_n L/2)$$

Here $Z_*$ is the impedance of the valve in a vacuum, normalized, like all the impedances henceforth, to the impedance $\rho c$, $\omega_0$ is the natural frequency of the valve in a vacuum, $m_0$ is the surface density of the valve, $Z_n$ is the normalized impedance of the interaction of the valve, situated in the cross-section $z = 0$, and the $n$th normal mode, and $Z_n^{\pm}$ is the normalized impedance of the cavity with rigid walls and two moving valves for the symmetric $n$th standing wave (superscript plus) and the for the antisymmetric $n$th standing wave (superscript minus).

The equation of motion of the valve, situated in the cross-section with coordinate $z = (j + 1)L$, has the form

$$(g - m\omega^2)U_{j+1} = \iint\limits_{G_2} (P_j(M,(j+1)L) - P_{j+1}(M,(j+1)L))dS \tag{3.3}$$

Using representation (2.10), relation (2.3) and formulae (2.11) and (2.12), we can rewrite the equation of motion of the valve (3.3) in the form

$$u_{j+1}(Z_* + \sum(Z_n^+ + Z_n^-)) = \frac{1}{2}(u_j + u_{j+1})\sum(Z_n^+ - Z_n^-) \tag{3.4}$$

For this second-order linear difference equation with constant coefficients it is necessary to solve a boundary-value problem with condition (3.3) for $j = 0$ and, taking into account the limiting absorption principle, the condition than $u_j$ decreases as $j \to \infty$.

We will seek the sequence of quantities $u_j$ in the form

$$u_j = \exp(ij\Gamma) \tag{3.5}$$

where $\Gamma$ is the propagation constant of the waves in the waveguide with a periodic system of obstacles.

To determine the propagation constant $\Gamma$ from Eq. (3.4), taking (3.5) into account, we have the dispersion equation

$$\cos\Gamma = W; \quad W = (Z_* + \sum(Z_n^+ + Z_n^-))(\sum(Z_n^+ + Z_n^-))^{-1} \tag{3.6}$$

Note that $W(\omega)$ is a real quantity for any valve of the frequency $\omega$.

Equation (3.6) has real roots when $|W(\omega)| < 1$, which defines frequency domains – passbands of the periodic system of obstacles in the waveguide, when similar obstacles are placed at equal distance $L$. Under these conditions, Eq. (3.6) has two simple real roots $\Gamma_{1,2} = \pm\arccos W$ in the interval $(-\pi, \pi)$, to which three correspond two linearly independent bounded solutions $u_j^{(1)} = \exp(ij\Gamma_1)$ and $u_j^{(2)} = \exp(ij\Gamma_2)$. In a waveguide with a semi-infinite set of obstacles, two harmonic processes correspond to them, which we will call travelling waves. The solution $u_j^{(1)}$ corresponds to a wave propagating in the positive direction of the $Oz$ axis, while $u_j^{(2)}$ corresponds to a wave propagating in the negative direction. If $|W(\omega)| = 1$, Eq. (3.6) has multiple roots. In this case, when $W = 1$, one solution of the difference equation has the form $u_j^{(1)} = u_0$, and the valves oscillate in phase, and when $W = -1$, the relations $u_j^{(1)} = (-1)^j u_0$ are satisfied and neighbouring valves oscillate in antiphase. The other linearly independent solution has the form $u_j^{(2)} = ju_j^{(1)}$ and is not bounded.

When $|W(\omega)| > 1$ in a waveguide with a periodic system of obstacles there are no propagating waves, since the roots of Eq. (3.6) are pure imaginary, corresponding to exponentially increasing or exponentially decreasing non-uniform waves.

After finding the value of the propagation constant $\Gamma$, the value of the displacement amplitude $u_0$ is found from Eq. (3.2), taking relation (3.5) into account,

$$u_0 = 2\eta_s\Big(Z_* + \sum Z_n + \frac{1}{2}\sum(Z_n^+ + Z_n^-) - \frac{1}{2}\exp(i\Gamma)\sum(Z_n^+ - Z_n^-)\Big)^{-1} \tag{3.7}$$

Using Euler's formula for the exponential function with imaginary exponent and Eq. (3.6), we can express the quantity $\exp(i\Gamma)$ in terms of the impedances of the system and we can write expression (3.7) in the form

$$u_0 = 2\eta_s/Z; \quad Z = Z_* + \sum Z_n + \tilde{Z}, \quad \tilde{Z} = (Z_*/2 + \sum Z_n^+)^{1/2}(Z_*/2 + \sum Z_n^-)^{1/2} - Z_*/2 \tag{3.8}$$

where $Z$ is the impedance of the semi-infinite system of barriers in the waveguide, $\sum Z_n$ is the impedance of the left channel of the waveguide $z < 0$, $\tilde{Z}$ is the impedance of the right channel of the semi-infinite system of barriers when $z > 0$.

The coefficient $r_{ns}$, according to Eq. (2.8), taking expression (3.8) into account, can be calculated from the formula

$$r_{ns} = \delta_{ns} - 2\sqrt{Z_n Z_s}/Z \tag{3.9}$$

Note that the choice of the normalization of the normal modes in relation (2.1) has enabled us to express the principle of duality in the simplest form – here there is a symmetry of the scattering matrix $r_{ns} = r_{sn}$.

## 4. THE REFLECTION OF A WAVE FROM ONE AND TWO BARRIERS

To analyse the results obtained above, we will obtain solutions of problems of the reflection of a wave of the type (2.4) from one and two barriers in a waveguide.

Suppose two similar barriers with valves are situated in the cross-section $z = 0$ and $z = L$. We will seek the acoustic pressures in the left part of the waveguide, where $-\infty < z < 0$, in the form of the expansion

$$P(M, z) = Q_0(M, z) + \sum r_{ns}^{(2)} p_s^-(M, z)$$

and in the right part of the waveguide where $z > L$ in the form

$$P(M, z) = \sum t_{ns}^{(2)} p_n^+(M, z - L)$$

where $r_{ss}^{(2)}$ and $t_{ss}^{(2)}$ are the reflection and transmission coefficients, and $r_{ns}^{(2)}$ and $t_{ns}^{(2)}$, when $n \neq s$, are transformation factors of the incident wave into the $n$th reflected and transmitted normal mode. For acoustic pressure between the barriers, where $0 < z < L$, we will have representation (2.9).

As when obtaining formula (2.8) we have $t_{ns} = \eta_n u_1$, where $u_1$ is the displacement of the valve situated in the cross-section $z = L$.

The final expressions for the reflection and transmission coefficients and for the transformation factor in the case of two barriers have the form

$$\begin{aligned} r_{ns}^{(2)} &= \delta_{ns} - 2\sqrt{Z_n Z_s}/Z^{(2)}, \quad t_{ns}^{(2)} = 2\sqrt{Z_n Z_s}/Z^{(2)} \\ Z^{(2)} &= Z_* + \sum Z_n + \hat{Z}, \quad \hat{Z} = ((Z_* + \sum Z_n)^{-1} + 2(\sum (Z_n^+ + Z_n^-))^{-1})^{-1} \end{aligned} \tag{4.1}$$

where $Z^{(2)}$ is the impedance of the system of two valves and the cavity between them in the waveguide, and $\hat{z}$ is the impedance of the system consisting of the cavity ($0 < z < L$), which is to the right of the first barrier ($z = 0$), the second barrier ($z = L$) and the semi-infinite channel of waveguide ($z > L$), situated to the right of the second barrier.

If there is only one barrier in the waveguide in the cross-section $z = 0$, the expressions for the reflection coefficient from one barrier ($r_{ss}^{(1)}$) the transmission coefficient ($t_{ss}^{(1)}$) and the transformation factors ($r_{ns}^{(1)}$ and $t_{ns}^{(1)}$) are obtained from formulae (4.1), by taking the limit as $L \to 0$ and halving the mass of the valve. We will have

$$r_{ns}^{(1)} = \delta_{ns} - 2\sqrt{Z_n Z_s}/Z^{(1)}, \quad t_{ns}^{(1)} = 2\sqrt{Z_n Z_s}/Z^{(1)}, \quad Z^{(1)} = Z_* + 2\sum Z_n \tag{4.2}$$

where $Z^{(1)}$ is the impedance of one valve in the waveguide.

The first expression of (4.2) can also be obtained if we assume, in the first formula of (4.1), that the impedances of the waveguide valves to the right and to the left of the barrier are similar and $\hat{Z} = \sum Z_n$. The displacement of the valve $u_0$ is then found from the first formula of (3.8), in which we must replace the impedance $Z$ by $Z^{(1)}$.

Note that, if we assume $Z^{(1)} = Z_* + \sum Z_n$ in the first formula of (4.2), we obtain an expression for the reflection coefficients of one barrier in the waveguide, on the right of which there is a vacuum in the waveguide.

## 5. DISCUSSION OF THE RESULTS

We will investigate how the reflection coefficients of a piston-type normal mode ($s = 0$) depend on the frequency of the incident field $\omega$.

In the case of a single barrier, it follows from the first of (4.2) that

$$r_{00}^{(1)} = (Z_* + 2(Z_1 + Z_2 + \dots))(Z_* + 2(Z_0 + Z_1 + Z_2 + \dots))^{-1} \tag{5.1}$$

As the frequency of the incident wave $\omega$ approaches the onset frequency $\tilde{\omega}_n$ of the $n$th waveguide mode ($n > 0$) we have $\gamma_n \to 0$ and $Z_n \to \infty$. By taking the limit in formula (5.1) we obtain that $\lim r_{00}^{(1)} = 1$ as $\omega \to \tilde{\omega}_n$.

The frequency of total transmission of the incident zeroth normal mode, on which $r_{00}^{(1)} = 0$, while $t_{00}^{(1)} = 1$, according to expression (5.1) is found from the equation $Z_* + 2(Z_1 + Z_2 + \ldots) = 0$. It always has a unique solution. First, the required frequency is less than the onset frequency of the first normal mode $\tilde{\omega}_1$, since under these conditions the impedances occurring in the equation are pure imaginary quantities. Second, it is less than the natural frequency of the valve $\omega_0$, since here, as the frequency increases from zero to the natural frequency of the valve, the value of $\mathrm{Im}(-Z_*)$ increases monotonically form $-\infty$ to zero, while the quantity $\mathrm{Im}(Z_1 + Z_2 + \ldots)$ decreases monotonically from zero to a certain finite value.

At the frequency of total transmission of the incident field $p_0^+(M, z)$ we have the relation $\sigma_2|u_0| = \sigma|u|$, where $u_0$ is the displacement of the valve and $u$ is the displacement of the fluid in the incident wave in the cross-section $z = 0$.

We will now consider the case when the valve completely covers the waveguide cross-section, i.e. the region $G_2$ coincides with the region $G$. The scattering by the valve of only the normal mode $p_0^+(M, z)$ is of interest, the other normal modes being reflected from the valve as from an absolutely rigid wall, since the average pressure of these modes on the valve is equal to zero and the piston is fixed. In the case when a normal mode $p_0^+(M, z)$ is incident on the barrier, only the mode $r_{00}^{(1)} p_0^-(M, z)$ is reflected. Formula (5.10) for the reflection coefficient can be rewritten in the form

$$r_{00}^{(2)} = 1 - 2Z_0/Z^{(2)} \tag{5.2}$$

whereas $r_{n0}^{(1)} = 0$ when $n > 0$. In this case the equality $r_{00}^{(1)} = 0$ is satisfied at the natural frequency of the valve $\omega_0$.

The reflection coefficient of a piston wave in the case when two similar valves completely block the waveguide channel, by relations (4.1) are given by the formula

$$r_{00}^{(2)} = 1 - 2Z_0/Z^{(2)}; \quad Z^{(2)} = Z_* + Z_0 + \hat{Z}, \quad \hat{Z} = ((Z_* + Z_n)^{-1} + (Z_n^+ + Z_n^+)^{-1})^{-1} \tag{5.3}$$

The transformation factors $r_{n0}^{(2)} = 0$ when $n = 1, 2, \ldots$ .

If follows from formula (5.3) that at the natural frequencies of a cavity with rigid walls and length $L$, defined by the formula $\omega_n' = \pi nc/L$, one of the impedances $Z_0^+$ or $Z_0^-$ is equal to zero, while the other is equal to infinity. At these frequencies the expression of the reflection coefficient takes the form

$$r_{00}^{(2)} = Z_*/(Z_* + Z_0) \tag{5.4}$$

A comparison of expressions (5.2) and (5.4) shows that at frequencies $\omega_n'$ the reflection coefficients of two valves with impedances $Z_*$ and from a single valve with an impedance $2Z_*$ are the same.

Another series of frequencies $\omega_n''$, at which the moduli of the reflection coefficients $r_{00}^{(1)}$ and $r_{00}^{(2)}$ are identical, is found from the equations

$$Z_*/2 + Z_0^+ = 0 \quad \text{or} \quad Z_*/2 + Z_0^- = 0 \tag{5.5}$$

Note that $r_{00}^{(2)} = 0$ at the natural frequency of the valve in a vacuum ($\omega = \omega_0$) and at frequencies $\omega_n'$, which are found from the equation

$$Z_* + Z_0^+ + Z_0^- = 0 \tag{5.6}$$

At sufficiently high frequencies, Eqs (5.5) and (5.6) can replaced by the following approximate equations respectively: $\mathrm{cth}(kL) = mk/(4\rho)$ and $\mathrm{cth}(kL) = mk/(2\rho)$. An analysis of the roots of the approximate equations shows that they satisfy the inequalities $\omega_n' < \omega_n^* < \omega_n''$.

We will begin an analysis of the frequency dependence of the reflection coefficient of a semi-infinite system of barriers with the case when the valves completely block the waveguide channel. We obtain the following expression for the reflection coefficient of a piston wave

$$r_{00} = 1 - 2Z_0/Z; \quad Z = Z_* + Z + \tilde{Z}, \quad \tilde{Z} = (Z_*/2 + Z_0^+)^{1/2}(Z_*/2 + Z_0^-)^{1/2} - Z_*/2 \tag{5.7}$$

where $Z$ is the impedance of a semi-infinite system of barriers, each of which completely blocks the waveguide channel.

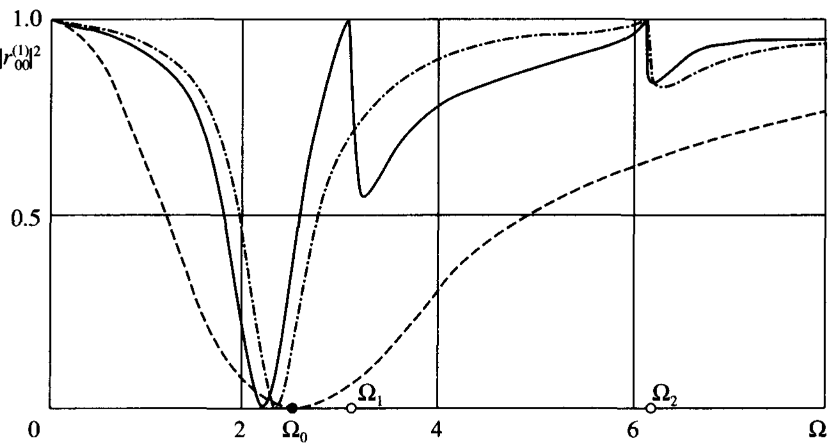


Fig. 2

It follows from formula (5.7) that, at a frequency of the incident wave $\omega = \omega_0$, equal to the natural frequency of the valve, the relations $\tilde{Z} = Z_0$, $Z = 2Z_0$ are satisfied and the reflection coefficient $r_{00}$ is equal to zero.

When the frequency of the incident field approaches the natural frequencies of the cavity, formed by neighbouring barriers in the waveguide, $r_{00} \to 1$.

If the valves completely block the waveguide cross-section, the expression for $W$ from the second formula of (3.6) takes the form

$$W = (Z_* + Z_0^+ + Z_0^-)/(Z_0^+ - Z_0^-)$$

The boundaries of the passband for a semi-infinite system of valves, which completely block the waveguide channel, are found from the equation $W = \pm 1$. These equations, in turn, lead to equations of the form (5.5), whence it follows that the boundaries of the passband, at which $r_{00} = 1$, correspond to the frequencies $\omega_n'$ and $\omega_n''$.

Numerical calculations of the squares of the moduli of the reflection coefficients as a function of the dimensionless frequency $\Omega = kH$ were carried out for the case of one, two and a semi-infinite set of barriers in a plane waveguide $0 < x < H, -\infty < z < \infty$. Assuming that the wave processes are independent of the variable $y$, we have: the region $G$ is a rectangle of height $H$ and unit width. A rectangular valve in that case occupies a rectangular part, such that $h_1 \le x \le h_2$. The eigenfunctions $\varphi_n(x)$ of the operator $\Delta_\perp = d^2/dx^2$, normalized in accordance with formula (2.2), which satisfy the boundary conditions $\varphi_n'(0) = \varphi_n'(H) = 0$, are given by the formula

$$\varphi_n(x) = \sqrt{2k/\varepsilon_n\gamma_n}\cos(\pi nx/H), \quad n = 0, 1, 2, \ldots; \quad \varepsilon_0 = 2, \quad \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \ldots = 1$$

while the eigenvalues

$$\mu_n = -(\pi n/H)^2$$

For the Fourier coefficients of the characteristic function of the section $[h_1, h_2]$ according to Eq. (2.7) we have

$$\eta_0 = \frac{h_2 - h_1}{H}, \quad \eta_n = \frac{\sqrt{2}}{\pi n}\left(\sin\frac{\pi nh_2}{H} - \sin\frac{\pi nh_1}{H}\right) \text{ for } n = 1, 2, \ldots$$

For the calculations we assumed that the dimensionless natural frequency of the valve in a vacuum $\Omega_0 = \omega_0 H/c = 2.5$ and the ratio $m/(\rho H) = 0.5$.

In Fig. 2 we show graphs of the square of the modulus of the reflection coefficients $r_{00}^{(1)}$ for one valve when it blocks different parts of the waveguide cross-section. Here the following dimensionless frequencies should be noted: the natural frequency of the valve $\Omega_0$ and the onset frequencies of the normal modes $\Omega_n = \pi n$. Even values of $n$ correspond to normal modes that are symmetric with respect to the waveguide axis, while odd values of $n$ correspond to normal modes that are antisymmetric with

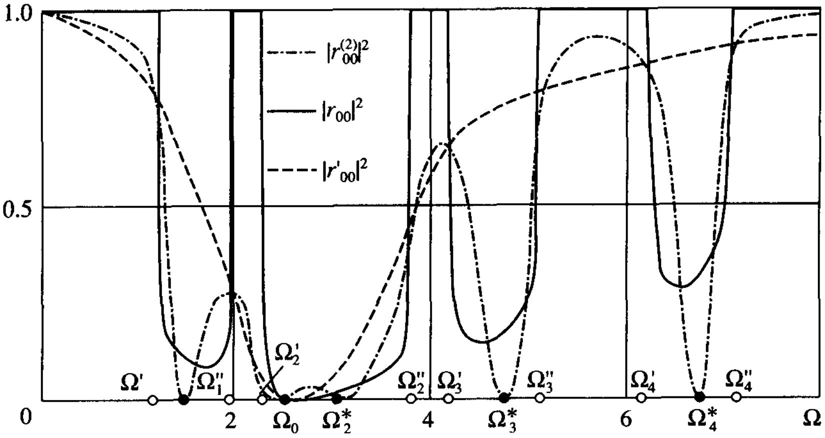


Fig. 3

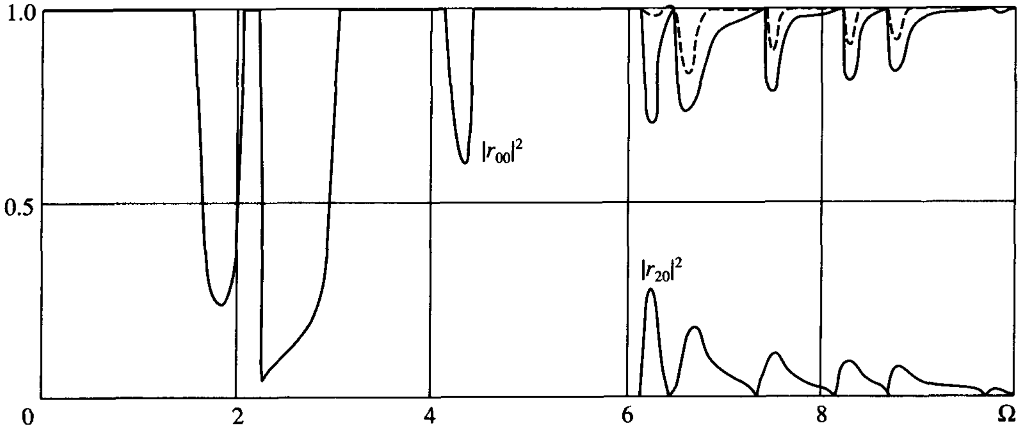


Fig. 4

respect to the waveguide axis. The dashed curves correspond to the case when the valve completely blocks the waveguide cross-section, the dash–dot curve corresponds to the case when the valve is situated at the centre of the waveguide and its width is half the width of the waveguide, while the continuous curve is for the case when the valve is in tight contact with the waveguide wall. When the valve surface is reduced and the surface of the rigid diaphragm is correspondingly increased, the modulus of the reflection coefficient increases, with the exception of a small frequency band in the region of $\Omega_0$. In a waveguide with a symmetrically placed valve in the diaphragm, only symmetrical normal modes are excited.

In Fig. 3 we show graphs of the frequency dependence of the square of the modulus of the reflection coefficient $r_{00}^{(2)}$ (the dash–dot curve) and $r_{00}$ (the continuous curve) for two and a semi-infinite system of similar barriers in a waveguide, respectively. We also show the frequency dependence of the square of the modulus of the reflection coefficient $r'_{00}$ of a single valve, but with twice the mass (the dashed curve), where $r'_{00} = Z_*/(Z_* + Z_0)$. In all these cases the valves completely block waveguide channel, and a distance between neighbouring barriers is chosen so that $L/H = 1.5$. Along the abscissa axis we have plotted the dimensionless frequencies $\Omega'_n = \omega'_n H/c$ and $\Omega''_n = \omega''_n H/c$. At these frequencies the values of the square of the moduli of $r_{00}^{(2)}$ and $r'_{00}$ are identical, and these frequencies are simultaneously the limits of the passband, where $|r_{00}| < 1$, and the stop band, where $|r_{00}| = 1$. We have denoted the dimensionless frequencies at which $r_{00}^{(2)} = 0$ by $\Omega^*_n = \omega^*_n H/c$.

Note that at sufficiently high frequencies the maxima of the modulus of the reflection coefficient for two barriers lie in the stop bands for the case of a semi-infinite set of barriers, while the minima lie in the passbands.

In Fig. 4 we show graphs of the square of the modulus of $r_{00}$ and $r_{20}$ for the case when the valves, occupying half the width of the waveguide, are situated in the middle of the barriers. To determine the frequencies in the stop band for a given reflection we note that, at these frequencies, it follows from the law of conservation of energy that the sum of the square of the modulus of the reflection coefficient of a travelling normal mode and of the squares of the moduli of the transformation factors of this mode in all the propagating normal modes is equal to unity. In the stop band, at frequencies less than the onset frequency of the first normal mode, the identity $|r_{00}| = 1$ is satisfied. At frequencies greater than the onset frequency of the first normal mode, but less than the onset frequency of the second normal mode, in the stop band the identity $|r_{00}|^2 + |r_{20}|^2 = 1$ is satisfied. The dashed curve in Fig. 4 shows the total reflected energy $W_* = |r_{00}|^2 + |r_{20}|^2$ as a function of $\Omega$ in the passbands.

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